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Fault tolerant control for systems with interval time-varying delay and actuator saturation

Zhou Gu[a,*], Dong Yue[b], Chen Peng[c], Jinliang Liu[d], Jianhua Zhang[e]

[a]College of Mechanical & Electronic Engineering, Nanjing Forestry University, Nanjing, China
[b]Department of Control Science and Engineering, Huazhong University of Science and Technology, Wuhan, China
[c]Institute of Information and Control Engineering Technology, Nanjing Normal University, Nanjing, China
[d]Department of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing, China
[e]Beijing Key Laboratory of Industrial Process Measurement & Control New Technology and System, North China Electric Power University, Beijing, China

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Abstract

This paper studies the problem of fault tolerant control design for time-delayed systems subject to actuator saturation. Delay- and fault-range-dependent estimate for the domain of attraction of the origin is presented using the linear matrix inequalities (LMIs) techniques. An illustrative example is exploited to show the effectiveness of the proposed design procedures.

1. Introduction

Systems with interval time-varying delay constitute basic mathematical models of real phenomena, for instance, chemical engineering systems, distributed networks, inferred grinding model, manual control, microwave oscillator, neural network, population dynamic model, ship stabilization, and systems with lossless transmission lines. The existence of time delay may cause instability or bad performances in dynamic systems. Hence the stability and stabilization problems for time-varying delay systems have received some attention [1–3].

*Corresponding author.
E-mail address: gzh1808@163.com (Z. Gu).

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Nearly all physical systems are subject to saturation constraints, such as actuator saturation and/or sensor saturation, which is usually a source of instability in control systems [4]. Considerable attention has been paid to such systems with saturation of control signals. In the existing literatures, there are mainly two developed approaches: (1) let saturation do not occur, which is called positive invariance approach [5] and (2) Allow saturation to take effect while guaranteeing asymptotic stability of the system [6,7]. Both of those two approaches, the main problem to be addressed is to get a large enough domain of initial states, which ensures an asymptotic stability of the system with input saturation.

Actuator/sensor failure is also inevitable in practical control applications. Much effort has been devoted to the fault tolerant control (FTC) design, since unexpected failure may result in a substantial damage of the system, and even be hazardous to the plant personnel and environments [8–11]. Therefore, FTC design is essential for the control system.

In this paper, we aim to develop a fault tolerant controller such that the system under the above instable sources at a same time, such as interval time-varying delay, uncertain actuator failures and actuator saturation, can be operating properly, since those phenomena are not isolated existence practically. Furthermore, an optimization problem with LMI constraints is formulated to obtain the largest contractively invariant set by using the proposed optimization algorithm. To the best of our knowledge, the problem remains open and challenging, which motivates us to the current study.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of real $n\times m$ matrices, I is the identity matrix of appropriate dimensions, $\|\cdot\|$ stands for the Euclidean vector norm or spectral norm as appropriate. The notation $X>0$ (respectively, $X<0$), for $X\in\mathbb{R}^{n\times n}$ means that the matrix X is a real symmetric positive definite (respectively, negative definite). $\mathcal{L}_{n,\tau_{21}}=\mathcal{L}([-\tau_2,0],\mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau_2 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The asterisk $*$ in a matrix is used to denote term that is induced by symmetry, Matrices, if they are not explicitly stated, are assumed to have compatible dimensions.

2. System description

Consider the following time-varying delay system with the saturation of control input

$$\dot{x}(t)=Ax(t)+A_dx(t-\tau(t))+B\sigma(u(t)) \tag{1}$$

$$x(t)=\phi(t),\quad t\in[-\tau_2,0] \tag{2}$$

where $x(t)\in\mathbb{R}^n$ and $u(t)\in\mathbb{R}^m$ denote the state and control vector, respectively. A, A_d and B are known constant matrices with appropriate dimensions, $\tau(t)$ is an interval time-varying delay which satisfies $0<\tau_1\le\tau(t)\le\tau_2$, $\phi(t)$ is a continuous vector valued initial function. The function $\sigma(\cdot)$ is the standard saturation function defined as follows:

$$\sigma(u(t))=[\sigma(u_1(t))\sigma(u_2(t))\cdots\sigma(u_m(t))]^T \tag{3}$$

where $\sigma(u_i(t))=\text{sign}(u_i(t))\min\{1,|u_i(t)|\}$.

The following actuator fault model and state-feedback control strategy are adopted:

$$u^F(t)=\Xi Kx(t) \tag{4}$$

where Ξ is an actuator fault scale factor matrix, and

$$\Xi=\text{diag}\{\xi_1,\xi_2\ldots\xi_m\},\quad 0\le\underline{\xi}_i\le\xi_i\le\overline{\xi}_i\ (i\in\mathcal{I}\triangleq\{1,2,\ldots,m\}) \tag{5}$$

where $\underline{\xi}_i$ and $\overline{\xi}_i$ $(i \in \mathcal{I})$ are given constants, especially, if $\xi_i = 0$ means that the ith actuator completely fails, and $\xi_i = 1$ denotes the ith actuator is normal.

Define

$$\Xi_0 = \mathrm{diag}[\xi_{10}, \ldots, \xi_{m0}], \quad \xi_{i0} = (\underline{\xi}_i + \overline{\xi}_i)/2 \tag{6}$$

$$\Xi_1 = \mathrm{diag}[\xi_{11}, \ldots, \xi_{m1}], \quad \xi_{i1} = (\overline{\xi}_i - \underline{\xi}_i)/2 \tag{7}$$

Then, the matrix Ξ can be rewritten as

$$\Xi = \Xi_0 + \Xi_1 \Delta J \tag{8}$$

where $\Delta J = \mathrm{diag}[j_1, \ldots, j_m]$, $|j_i| < 1, i = 1, \ldots, m$.

The dynamic Eq. (1) with consideration of actuator fault model (4) is then described by

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau(t)) + B\sigma(\Xi K x(t)) \tag{9}$$

To estimate the domain of attraction, the following two sets are introduced.

$$\mathcal{L}(F) \triangleq \{x(t) \in \mathbb{R}^n : |f_i x(t)| \le u_i, i \in \mathcal{I}\} \tag{10}$$

$$\mathcal{E}(P,1) \triangleq \{x(t) \in \mathbb{R}^n : x(t)^T P x(t) \le 1\} \tag{11}$$

where f_i is the ith row of the matrix $F \in \mathbb{R}^{n\times n}$, and $P \in \mathbb{R}^{n\times n}$ is a positive-definite matrix.

We introduce the following definition and lemmas firstly, which will be used in the subsequent development.

Definition 1 (*Zhang et al. [12]*). For an initial condition $x_0 \in \mathcal{L}_{n,\tau_2}$, suppose the state trajectory of system (1) $x(t,x_0)$ is asymptotically stable, then the domain of attraction of the origin is

$$\mathcal{X} := \left\{ x_0 \in \mathcal{L}_{n,\tau_2} : \lim_{t\to\infty} x(t,t_0) = 0 \right\} \tag{12}$$

Lemma 1 (*Wang et al. [13]*). *Let U, V, W, X be real matrices of appropriate dimensions with X satisfying $X = X^T$. Then*

$$X + UVW + W^T V^T U^T < 0 \quad \textit{for all } V^T V \le I$$

if and only if there exists a scalar $\varepsilon > 0$ such that

$$X + \varepsilon U U^T + \varepsilon^{-1} W^T W < 0$$

Lemma 2 (*Gu et al. [14]*). *For any constant matrix $R \in \mathbb{R}^{n\times n}$, $R > 0$, scalars $\tau_m \le \tau(t) \le \tau_M$, and vector function $\dot{x} : [-\tau_m, 0] \to \mathbb{R}^n$ such that the following integration is well defined, it holds that*

$$-\tau_m \int_{t-\tau_m}^{t} \dot{x}^T(t) R \dot{x}(t) \le \begin{bmatrix} x(t) \\ x(t-\tau_m) \end{bmatrix}^T \begin{bmatrix} -R & R \\ * & -R \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau_m) \end{bmatrix} \tag{13}$$

Lemma 3 (*Park and Wan Ko [15]*). *Suppose S, T, and Ω are constant matrices of appropriate dimensions. Then*

$$(\tau(t) - \tau_m) S + (\tau_M - \tau(t)) T + \Omega < 0 \tag{14}$$

is true for any $\tau(t) \in [\tau_m, \tau_M]$ *if and only if*

$$(\tau_M - \tau_m)S + \Omega < 0 \tag{15}$$

$$(\tau_M - \tau_m)T + \Omega < 0 \tag{16}$$

Let $\mathcal{W}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are 1 or 0. W_i $(i = 1, 2, \ldots, 2^m)$ is the element of $\mathcal{W}$, and define $W_i^- = I - W_i$, obviously, W_i is also an element of $\mathcal{W}$.

Lemma 4 (*Hu et al.* [7]). *Let* $F, H \in \mathbb{R}^{m \times n}$ *be given. For* $x \in \mathbb{R}^n$, *if* $x \in \mathcal{L}(H)$, *then*

$$\sigma(Fx(t)) \in co\{W_i Fx(t) + W_i^- Hx(t) : i \in \mathcal{I}\} \tag{17}$$

where the notation $co\{\cdot\}$ *denotes the convex hull of a set.*

The objective of the paper is to design a reliable state-feedback controller for the system (1) with consideration of the actuators saturation and failures such that the closed-loop system (9) is asymptotically stable.

3. Main result

In this section, we will first focus on the condition of local asymptotic stability for the systems with the actuator saturation and failures. The design of reliable controller is then presented.

Theorem 1. *For given scalars* τ_1, τ_2, *the closed-loop system* (9) *with consideration of all possible faults is asymptotically stable within the set* $\mathcal{E}(P,1)\{x \in \mathbb{R}^n | x^T(t)Px(t) \le 1\}$, *if there exist matrices* $P > 0$, $R_1 > 0$, $R_2 > 0$, $Q_1 > 0$, $Q_2 > 0$ *and scalars* ε *such that the following matrix inequalities hold*

$$\mathcal{E}(P,1) \subset \mathcal{L}(H) \tag{18}$$

$$\begin{bmatrix} \Psi_j^0 & * & * & * & * \\ P\overline{\mathcal{A}} & -PQ^{-1}P & * & * & * \\ \Phi(l) & 0 & -\sqrt{\tau_{21}}Q_2 & * & * \\ \varepsilon\Psi_j^1 & \varepsilon\Xi_1^T W_j^T B^T P & 0 & -\varepsilon I & * \\ \Psi_j^2 & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0 \quad \begin{pmatrix} l = 1,2 \\ j = 1,2,\ldots,2^m \end{pmatrix} \tag{19}$$

where

$$\Psi_j^0 = \begin{bmatrix} \Psi_{11} & * & * & * \\ \Psi_{21} & \Psi_{22} & * & * \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & * \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{bmatrix}$$

$$\Phi(1) = S^T, \quad \Phi(2) = T^T$$

$$\Psi_j^1 = [\Xi_1^T W_j^T B^T P \;\; 0 \;\; 0 \;\; 0], \quad \Psi_j^2 = [K \;\; 0 \;\; 0 \;\; 0]$$

$$\Psi_{11}=R_1+R_2-Q_1+P(A+B(W_j\Xi_0K+W_j^-H))+(A+B(W_j\Xi_0K+W_j^-H))^TP$$
$$\Psi_{21}=Q_1+S_1^T,\quad \Psi_{22}=-R_1-Q_1+S_2+S_2^T$$
$$\Psi_{31}=T_1^T-S_1^T+A_d^TP,\quad \Psi_{32}=S_3+T_2^T-S_2^T,\quad \Psi_{33}=T_3+T_3^T-S_3-S_3^T$$
$$\Psi_{41}=-T_1^T,\quad \Psi_{42}=S_4-T_2^T,\quad \Psi_{43}=T_4-S_4-T_3^T,\quad \Psi_{44}=-R_2-T_4-T_4^T$$
$$\overline{\mathcal{A}}=[A+B(W_j\Xi_0K+W_j^-H)\ \ 0\ \ A_d\ \ 0],\quad \tau_{21}=\tau_2-\tau_1$$

Proof. Choose the following Lyapunov function for the system (9) as

$$V(x_t)=\sum_{i=1}^{3}V_i(x_t) \tag{20}$$

where

$$V_1(x_t)=x^T(t)Px(t)$$
$$V_2(x_t)=\int_{t-\tau_1}^{t}x^T(s)R_1x(s)\,ds+\int_{t-\tau_2}^{t}x^T(s)R_2x(s)\,ds$$
$$V_3(x_t)=\tau_1\int_{-\tau_1}^{0}\int_{t+s}^{t}\dot{x}^T(v)Q_1\dot{x}(v)\,dv\,ds+\int_{-\tau_2}^{-\tau_1}\int_{t+s}^{t}\dot{x}^T(v)Q_2\dot{x}(v)\,dv\,ds$$

Taking the derivative of $V(x_t)$ along the trajectory of the system (9), and using Lemma 2, It follows:

$$\dot{V}_1(x_t)=2x^T(t)P\dot{x}(t)$$
$$\dot{V}_2(x_t)=x^T(t)(R_1+R_2)x(t)-\sum_{i=1}^{2}x^T(t-\tau_i)R_ix(t-\tau_i)$$
$$\dot{V}_3(x_t)\le\dot{x}^T(t)[\tau_1^2Q_1+\tau_{21}Q_2]\dot{x}(t)+\begin{bmatrix}x(t)\\x(t-\tau_1)\end{bmatrix}^T\begin{bmatrix}-Q_1 & *\\ Q_1 & -Q_1\end{bmatrix}\begin{bmatrix}x(t)\\x(t-\tau_1)\end{bmatrix}$$
$$-\int_{t-\tau_2}^{t-\tau_1}\dot{x}^T(s)Q_2\dot{x}(s)\,ds$$

Employing the free-weighting matrix method [16,1], we have

$$2\zeta^T(t)S\left[x(t-\tau_1)-x(t-\tau(t))-\int_{t-\tau(t)}^{t-\tau_1}\dot{x}(s)\,ds\right]=0 \tag{21}$$

$$2\zeta^T(t)T\left[x(t-\tau(t))-x(t-\tau_2)-\int_{t-\tau_2}^{t-\tau(t)}\dot{x}(s)\,ds\right]=0 \tag{22}$$

where $\zeta(t)=[x^T(t)\ \ x^T(t-\tau_1)\ \ x^T(t-\tau(t))\ \ x^T(t-\tau_2)]^T$.

Similar the method in [17,18], we have

$$-2\zeta^T(t)S\int_{t-\tau(t)}^{t-\tau_1}\dot{x}(s)\,ds\le(\tau(t)-\tau_1)\zeta^T(t)Q_2^{-1}S^T\zeta(t)+\int_{t-\tau(t)}^{t-\tau_1}\dot{x}^T(s)Q_2\dot{x}(s)\,ds \tag{23}$$

$$-2\zeta^T(t)T\int_{t-\tau_2}^{t-\tau(t)}\dot{x}(s)\,ds \le (\tau_2-\tau(t))\zeta^T(t)TQ_2^{-1}T^T\zeta(t)+\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)Q_2\dot{x}(s)\,ds \tag{24}$$

From Eq. (18) and Lemma 4, we can rewritten $\sigma(u(t))$ for all $x(t)\in\mathcal{E}(P,1)$ as

$$\sigma(\Xi Kx(t)) = K_j x(t) \tag{25}$$

where

$$K_j=\sum_{j=1}^{2^m}\theta_j(W_j\Xi K+W_j^- H)\quad\left(\sum_{j=1}^{2^m}\theta_j=1,\theta_j>0\right) \tag{26}$$

Then, the closed-loop system (9) can be further rewritten as

$$\dot{x}(t)=Ax(t)+A_d x(t-\tau(t))+BK_j x(t) \tag{27}$$

Calculate the time derivatives of $V(x_t)$ along the trajectories of the system (27), it yields

$$\begin{aligned}
&\dot{V}(x_t)\le 2x^T(t)P\dot{x}(t)+x^T(t)(R_1+R_2)x(t)-\sum_{i=1}^{2}x^T(t-\tau_i)R_i x(t-\tau_i)\\
&+\dot{x}^T(t)\mathcal{Q}\dot{x}(t)+\begin{bmatrix}x(t)\\ x(t-\tau_1)\end{bmatrix}^T\begin{bmatrix}-Q_1 & * \\ Q_1 & -Q_1\end{bmatrix}\begin{bmatrix}x(t)\\ x(t-\tau_1)\end{bmatrix}\\
&+2\zeta^T(t)[Sx(t-\tau_1)+(T-S)x(t-\tau(t))-Tx(t-\tau_2)]\\
&+2\zeta^T(t)S[(A+BK_j)x(t)+A_d x(t-\tau(t))-\dot{x}(t)]\\
&+\zeta^T(t)[(\tau(t)-\tau_1)SQ_2^{-1}S^T+(\tau_2-\tau(t))TQ_2^{-1}T^T]\zeta(t)\\
&\le\sum_{j=1}^{2^m}\theta_j\zeta^T(t)\{\Psi_j+[(\tau(t)-\tau_1)SQ_2^{-1}S^T+(\tau_2-\tau(t))TQ_2^{-1}T^T]\}\zeta(t)
\end{aligned}$$

where $\Psi_j=\Psi_j^0+\Psi_j^{1^T}\Delta J\Psi_j^2+\Psi_j^{2^T}\Delta J^T\Psi_j^1+\overline{\mathcal{A}}^T\mathcal{Q}\overline{\mathcal{A}}$, $\mathcal{Q}=\tau_1^2Q_1+\tau_{21}Q_2$, then one can conclude that if

$$\Psi_j+[(\tau(t)-\tau_1)SQ_2^{-1}S^T+(\tau_2-\tau(t))TQ_2^{-1}T^T]<0\quad j=1,2,\ldots,2^m \tag{28}$$

holds, then $\dot{V}(x_t)<0$.

Based on Lemmas 1 and 3 and Schur complement, we can know that Eq. (19) is a sufficient condition to guarantee Eq. (28) holds. This completes the proof. □

Theorem 1 gives a sufficient condition to guarantee the stability of the closed-loop system (9). Now we will give an LMI-based optimization algorithm to obtain the largest contractively invariant ellipsoid $\mathcal{E}(P,1)$ for the systems (27).

With the optimization method in [7], an exact invariant set with least degree of conservativeness can be formulated as

$$\begin{aligned}&\max\quad \alpha\\ &\text{s.t.}\quad\begin{cases}\alpha\Omega\subset\mathcal{E}(P,1)\\ (18)-(19)\end{cases}\end{aligned} \tag{29}$$

where $\Omega=\mathcal{E}(\Pi,1),\Pi\in\mathbb{R}^{n\times n}$.

Theorem 2. *For given scalars* τ_1, τ_2, ϵ *and matrix* $\Xi_i (i=0,1)$. *The closed-loop system* (9), *under the all possible faults and saturation of the control input*, *is asymptotically stable*, *if there exist matrices* $X>0$, $\overline{R}_1>0$, $\overline{R}_2>0$, $\overline{Q}_1>0$, $\overline{Q}_2>0$, Y *and scalars* $\varepsilon>0$ *such that the following LMIs hold*

$$\begin{bmatrix} u_i & * \\ l_i^T & u_i X \end{bmatrix} > 0 \quad i \in \mathcal{I} \tag{30}$$

$$\begin{bmatrix} \overline{\Psi}_j^0 & * & * & * & * \\ \overline{\Psi}_j^1 & -2\epsilon X + \epsilon^2 \overline{\mathcal{Q}} & * & * & * \\ \overline{\Phi}(l) & 0 & -\sqrt{\tau_{21}}\overline{Q}_2 & * & * \\ \varepsilon \overline{\Psi}_j^1 & \Xi_1^T W_j^T B^T & 0 & -\varepsilon I & * \\ \overline{\Psi}_j^2 & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0, \quad \begin{pmatrix} l=1,2 \\ j=1,2,\ldots,2^m \end{pmatrix} \tag{31}$$

where

$$\overline{\Psi}_j^0 = \begin{bmatrix} \overline{\Psi}_{11} & * & * & * \\ \overline{\Psi}_{21} & \overline{\Psi}_{22} & * & * \\ \overline{\Psi}_{31} & \overline{\Psi}_{32} & \overline{\Psi}_{33} & * \\ \overline{\Psi}_{41} & \overline{\Psi}_{42} & \overline{\Psi}_{43} & \overline{\Psi}_{44} \end{bmatrix}$$

$$\overline{\Psi}_j^1 = [AX + BW_i \Xi_0 Y + BW_i^- L \;\; 0 \;\; A + dX \;\; 0]$$

$$\overline{\Psi}_j^1 = [\Xi_1^T W_j^T B^T \;\; 0 \;\; 0 \;\; 0]$$
$$\overline{\Psi}_j^2 = [Y \;\; 0 \;\; 0 \;\; 0]$$
$$\overline{\Psi}_{11} = \overline{R}_1 + \overline{R}_2 - \overline{Q}_1 + AX + XA^T + BW_j \Xi_0 Y + \Xi_0^T W_j^T B^T + BW_j^- L + L^T W_j^{-T} B^T$$
$$\overline{\Psi}_{21} = \overline{Q}_1 + \overline{S}_1^T, \overline{\Psi}_{22} = -\overline{R}_1 - \overline{Q}_1 + \overline{S}_2 + \overline{S}_2^T$$
$$\overline{\Psi}_{31} = \overline{T}_1^T - \overline{S}_1^T + XA_d^T, \quad \overline{\Psi}_{32} = \overline{S}_3 + \overline{T}_2^T - \overline{S}_2^T, \quad \overline{\Psi}_{33} = \overline{T}_3 + \overline{T}_3^T - \overline{S}_3 - \overline{S}_3^T$$
$$\overline{\Psi}_{41} = -\overline{T}_1^T, \quad \overline{\Psi}_{42} = \overline{S}_4 - \overline{T}_2^T, \quad \overline{\Psi}_{43} = \overline{T}_4 - \overline{S}_4 - \overline{T}_3^T, \quad \overline{\Psi}_{44} = -\overline{R}_2 - \overline{T}_4 - \overline{T}_4^T$$
$$\overline{\Phi}(1) = \overline{S}^T, \quad \overline{\Phi}(2) = \overline{T}^T, \quad \overline{\mathcal{Q}} = \tau_1^2 \overline{Q}_1 + \tau_{21} \overline{Q}_2$$

In addition, *the gain of the fault tolerant controller in Eq.* (4) *is given by* $K = YX^{-1}$.

Proof. From [7], the constraint (18) is equivalent to

$$\begin{bmatrix} u_i & * \\ h_i^T & u_i P \end{bmatrix} \geq 0, \quad i \in \mathcal{I} \tag{32}$$

where h_i is the ith row of H.

Define $X = P^{-1}$ and $l_i = h_{iX}$, one can see that Eq. (32) is equivalent to Eq. (30) using Shur complement.

Due to $(\mathcal{Q} - \epsilon^{-1}P)\mathcal{Q}^{-1}(\mathcal{Q} - \epsilon^{-1}P) \geq 0$ for $\mathcal{Q} > 0, P > 0$ and $\epsilon > 0$, it gives that

$$-P\mathcal{Q}^{-1}P \leq -2\epsilon P + \epsilon^2 \mathcal{Q} \tag{33}$$

Obviously,

$$\begin{bmatrix} \Psi_j^0 & * & * & * & * \\ P\overline{\mathcal{A}} & -2\epsilon P_i + \epsilon^2 \mathcal{Q} & * & * & * \\ \Phi(l) & 0 & -\sqrt{\tau_{21}}Q_2 & * & * \\ \varepsilon \Psi_j^1 & \varepsilon \Xi_1^T W_j^T B^T P & 0 & -\varepsilon I & * \\ \Psi_j^2 & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0 \quad \begin{pmatrix} l = 1,2 \\ j = 1,2,\ldots,2^m \end{pmatrix} \tag{34}$$

is a sufficient condition of Eq. (19). Pre- and post-multiply Eq. (34) by diag$\{X,X,X,X,X,X,I,I\}$, it follows Eq. (31) holds. This completes the proof. □

To find an exact invariant set with least degree of conservativeness, Theorem 2 can be formulated as the following optimizing problem:

$$\begin{aligned} &\max \quad \alpha \\ &\text{s.t.} \quad \begin{cases} \alpha\Omega \subset \mathcal{E}(P,1) \\ (30){-}(31) \end{cases} \end{aligned} \tag{35}$$

where $\Omega = \mathcal{E}(\Pi,1), \Pi \in \mathbb{R}^{n \times n}$. Using Schur complement, one can know that Eq. (35) is equivalent to

$$\begin{aligned} &\inf \quad \gamma \\ &\text{s.t.} \quad \begin{cases} \begin{bmatrix} \gamma\Pi & * \\ I & X \end{bmatrix} \geq 0 \\ (30){-}(31) \end{cases} \end{aligned} \tag{36}$$

where $\gamma = 1/\alpha^2$.

Choose $\Xi_0 = I$ and $\Xi_1 = 0$ in Theorem 2, the system (27) is then becomes a normal system, i.e. there is no any failure occurring in the system's running process. The following corollary can be obtained using the similar method.

Corollary 1. *For given scalars τ_1,τ_2, the closed-loop system (9) under the constrain of saturation of control input is asymptotically stable, if there exist matrices $X > 0$, $\tilde{R}_1 > 0$, $\tilde{R}_2 > 0$, $\tilde{Q}_1 > 0$, $\tilde{Q}_2 > 0$ and scalars $\varepsilon > 0$ such that the following LMIs hold*

$$\begin{bmatrix} u_i & * \\ l_i^T & u_i X \end{bmatrix} > 0, \quad i \in \mathcal{I} \tag{37}$$

$$\begin{bmatrix} \tilde{\Psi}_j^0 & * & * \\ \tilde{\Psi}_j^1 & -2\epsilon X + \epsilon^2 \tilde{\mathcal{Q}} & * \\ \tilde{\Phi}(l) & 0 & -\sqrt{\tau_{21}}\tilde{Q}_2 \end{bmatrix} \tag{38}$$

where

$$\tilde{\Psi}_j^0 = \begin{bmatrix} \tilde{\Psi}_{11} & * & * & * \\ \tilde{\Psi}_{21} & \tilde{\Psi}_{22} & * & * \\ \tilde{\Psi}_{31} & \tilde{\Psi}_{32} & \tilde{\Psi}_{33} & * \\ \tilde{\Psi}_{41} & \tilde{\Psi}_{42} & \tilde{\Psi}_{43} & \tilde{\Psi}_{44} \end{bmatrix}$$

$$\tilde{\Psi}_j^1 = [AX + BW_i Y + BW_i^- L \ \ 0 \ \ A + dX \ \ 0]$$

$$\tilde{\Psi}_j^1 = [\Xi_1^T W_j^T B^T \ \ 0 \ \ 0 \ \ 0]$$

$$\tilde{\Psi}_j^2 = [Y \ \ 0 \ \ 0 \ \ 0]$$

$$\tilde{\Psi}_{11} = \tilde{R}_1 + \tilde{R}_2 - \tilde{Q}_1 + AX + XA^T + BW_j Y + W_j^T B^T + BW_j^- L + L^T W_j^{-T} B^T$$

$$\tilde{\Psi}_{21} = \tilde{Q}_1 + \tilde{S}_1^T, \quad \tilde{\Psi}_{22} = -\tilde{R}_1 - \tilde{Q}_1 + \tilde{S}_2 + \tilde{S}_2^T$$

$$\tilde{\Psi}_{31} = \tilde{T}_1^T - \tilde{S}_1^T + XA_d^T, \quad \tilde{\Psi}_{32} = \tilde{S}_3 + \tilde{T}_2^T - \tilde{S}_2^T, \quad \tilde{\Psi}_{33} = \tilde{T}_3 + \tilde{T}_3^T - \tilde{S}_3 - \tilde{S}_3^T$$

$$\tilde{\Psi}_{41} = -\tilde{T}_1^T, \quad \tilde{\Psi}_{42} = \tilde{S}_4 - \tilde{T}_2^T, \quad \tilde{\Psi}_{43} = \tilde{T}_4 - \tilde{S}_4 - \tilde{T}_3^T, \quad \tilde{\Psi}_{44} = -\tilde{R}_2 - \tilde{T}_4 - \tilde{T}_4^T$$

$$\tilde{\Phi}(1) = \tilde{S}^T, \tilde{\Phi}(2) = \tilde{T}^T, \quad \tilde{\mathcal{Q}} = \tau_1^2 \tilde{Q}_1 + \tau_{21}\tilde{Q}_2$$

Using the similar optimization method, one can get the largest "contractively invariant ellipsoid" as follows:

$$\begin{aligned} &\min \quad \tilde{\gamma} \\ &\text{s.t.} \quad \begin{cases} \begin{bmatrix} \tilde{\gamma}\tilde{\Pi} & * \\ I & X \end{bmatrix} \geq 0 \\ (37)-(38) \end{cases} \end{aligned} \tag{39}$$

where $\tilde{\Pi} \in \mathbb{R}^{n \times n}$.

4. A numerical example

In this section, a numerical example is presented to demonstrate the effectiveness of the proposed method. Attention is focused on the controller design for the delay system with both actuator failure and its saturation.

Example 1. The parameters of continuous-time system (1) with state time-delay satisfying $0 \leq \tau(t) \leq 0.35$ are given as

$$A = \begin{bmatrix} 0.5 & -1 \\ 0.5 & -0.5 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.6 & 0.4 \\ 0 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u_i = 5$$

The following two cases are considered:
Case i: The actuator is normal, i.e. $\Xi \equiv I_{m\times m}$;
Case ii: The actuator is abnormal, and the fault range is assumed that $0.4 \le \xi_1 \le 0.8$.
Corollary 1 is used to address the problem of Case 1, from which the following results can be obtained with $\epsilon = 1$:

$$P = \begin{bmatrix} 0.0962 & -0.0167 \\ -0.0167 & 0.0196 \end{bmatrix}$$
$$K = [-1.8463 \;\; 0.6562]$$
$$H = [-1.4684 \;\; 0.4640]$$

Three ellipsoids are plotted in Fig. 1, where the inner dotted-dash ellipsoid is obtained by the method of [19] whose system is in a normal condition, and the outer solid and dashed ellipsoid are the sets $\mathcal{E}(P,1)$ and αX_R under the condition of Case i, respectively, from which one can see clearly that the state of the examined system converges to the origin within the estimated domain of attraction despite actuator saturation and the interval time-varying delays. In addition, it can be shown that our approach gives a larger estimation of the domain of attraction then the existed one. Also it can be observed that the ellipsoid set is contained inside the set of admissible saturations $\mathcal{L}(H)$ as well.

Case ii investigates the problem of the control system subject to actuator failure, the reliable controller and its corresponding parameters can be obtained Theorem 2

$$P = \begin{bmatrix} 0.2312 & -0.0472 \\ -0.0472 & 0.0376 \end{bmatrix}$$
$$K = [-4.1995 \;\; 1.3613]$$
$$H = [-2.2108 \;\; 0.7414]$$

Fig. 2 shows the resulting invariant ellipsoids of the system with normal (the outer) and actuator fault (the inner) conditions, respectively. Obviously, the estimated ellipsoid under

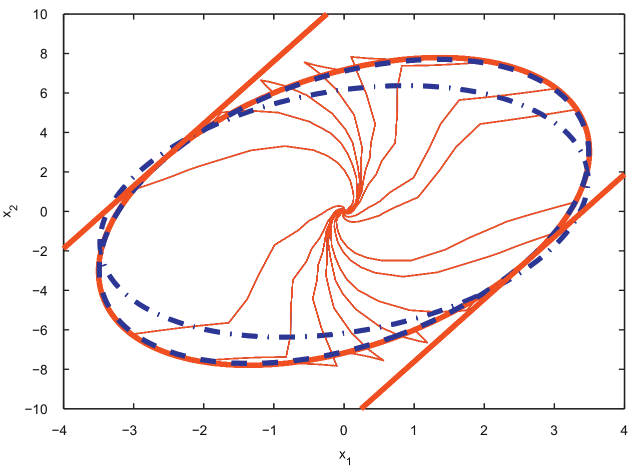

Fig. 1. Estimates of the domain of attraction and state trajectories without actuator failure.

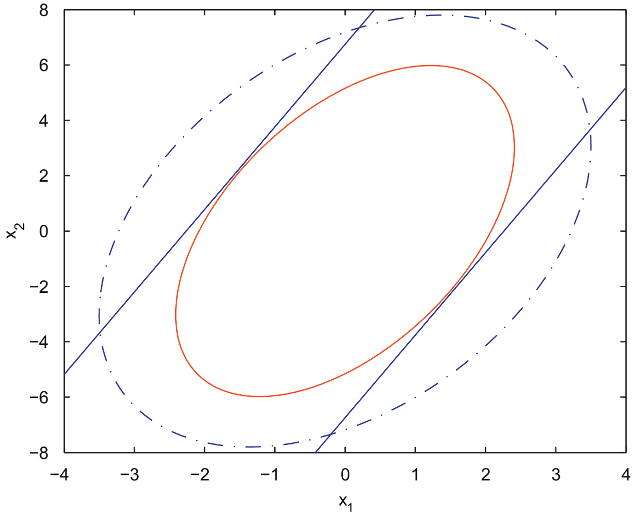

Fig. 2. Estimates of the domain of attraction under actuator failure.

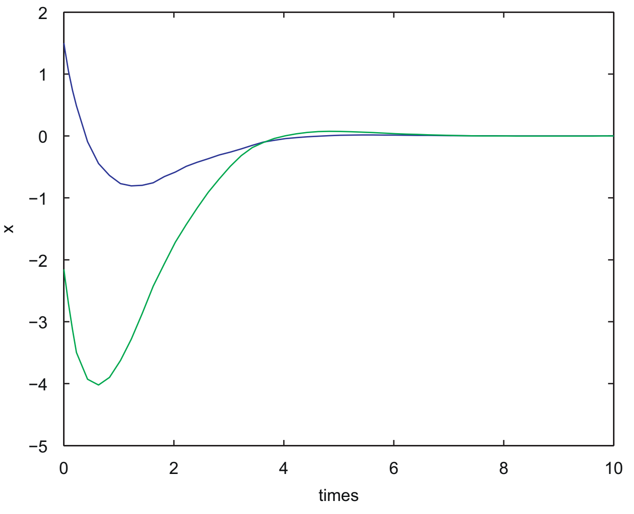

Fig. 3. State trajectories using fault-tolerant controller.

Case i is bigger than the one under Case ii, since the failure occurs at the actuator of the control system. From Fig. 3, one can see that the closed-loop system has a good control performances by using the proposed fault tolerant controller, although there exist some instable sources, such as time delay, actuator saturation, and failures etc.

5. Conclusion

The reliable control design for a class of interval time-varying delay systems subject to actuator failure and saturation is presented in this paper. Delay- and fault-range-dependent optimization approach is used to enlarge the estimation of the domain of attraction by a set of LMIs. A numerical example is used to show the effectiveness of the proposed method. However, the results in this paper are not concerned with the nondeterministic systems, such as switch system [1,20], which is the future research direction.

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